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EMBEDDINGS OF $C(\Delta)$ AND $L^1[0,1]$ IN BANACH LATTICES

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ABSTRACT

It is proved that if E is a separable Banach lattice with E' weakly sequentially complete, F is a Banach space and $T: E \to F$ is a bounded linear operator with $T'F'$ non-separable, then there is a subspace G of E, isomorphic to $C(\Delta)$, such that $T_{|G}$ is an isomorphism, where $C(\Delta)$ denotes the space of continuous real valued functions on the Cantor discontinuum. This generalizes an earlier result of the second-named author. A number of conditions are proved equivalent for a Banach lattice E to contain a subspace order isomorphic to $C(\Delta)$. Among them are the following: L^1 is lattice isomorphic to a sublattice of E'; $C(\Delta)'$ is lattice isomorphic to a sublattice of E'; E contains an order bounded sequence with no weak Cauchy subsequence; E has a separable closed sublattice F such that F' does not have a weak order unit.

§1. Let Δ denote the Cantor discontinuum and let L^1 denote the Banach lattice $L^1(m)$ where m is Lebesgue measure on the unit interval with respect to the σ-algebra of Lebesgue measurable sets. Our main results are as follows:

THEOREM 1. *Let E be a separable Banach lattice with E' weakly sequentially complete, let F be a Banach space, and let $T: E \to F$ be an operator with $T'F'$ non-separable. Then there is a complemented subspace $G \subset E$ isomorphic to $C(\Delta)$ such that $T_{|G}$ is an isomorphism.*

We note that Theorem 1 generalizes the main result of [10], where the special case of Theorem 1 was established for $E = C(X)$, X a compact metric space. The proof follows readily from lattice analogues of the results and techniques of [10]. The argument is given at the beginning of §4 and uses Lemmas 1, 2 and 3 of §3.

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THEOREM 2. *Let E be a Banach lattice. Then the following assertions are equivalent:*

(a) *L^1 is lattice isomorphic to a closed sublattice of E'.*

(b) *$C(\Delta)'$ is lattice isomorphic to a closed sublattice of E'.*

(c) *There is a compact space K and a continuous surjection $\varphi: K \to \Delta$ such that the corresponding isometry $C(\Delta) \to C(K)$ (defined by $f \to f \circ \varphi$) factors through E, $C(\Delta) \xrightarrow{T} E \xrightarrow{S} C(K)$ with T an order isomorphism and S positive. (If $0 < \varepsilon$, then K, φ, T, and S can be chosen so that $\|T\| \|S\| \leqq 1 + \varepsilon$.)*

(d) *There is a positive embedding $T: C(\Delta) \to E$.*

(e) *There is $0 < x \in E$ such that the order interval $[0, x]$ is not weakly sequentially precompact. (In other words, E has an order bounded sequence with no weak Cauchy subsequence.)*

(f) *There is a (positive) embedding $T: l^1 \to E$ with T majorizing.*

(g) *There is a separable closed sublattice F of E such that F' does not have a weak order unit.*

(h) *There is a separable closed sublattice F of E such that $l^1(\Gamma)$ for some uncountable set Γ is lattice isomorphic to a closed sublattice of F'.*

(i) *There is a (separable) closed sublattice F of E and an almost interval preserving operator T from F onto $C(\Delta)$.*

If, in addition, E is separable one can choose $F = E$ in (g), (h), *and* (i). *Moreover,* (a)–(i) *are then equivalent to*

(j) *Same as* (c) *with φ a homeomorphism. In particular, $C(\Delta)$ is order isomorphic to a closed subspace of E which is the range of a positive projection.*

We note that the equivalent statements (a), (b), (f), (h) and (i) of Theorem 2 are lattice analogues of results in general Banach spaces due to Pelczynski [8] and Hagler [3]. However, (d) has no analogue for general Banach spaces; for example $C(\Delta)$ does not embed in l^1 yet $C(\Delta)'$ embeds in $(l^1)' = l^\infty$.

COROLLARY. *If l^1 embeds in a Banach lattice E but not complementably then* (a)–(i) *of Theorem* 2 *hold.*

PROOF. If l^1 embeds in E then by theorem 2 of [7] it follows that c_0 or L^1 is lattice isomorphic to a closed sublattice of E'. But since l^1 is not complemented in E, c_0 does not embed in E' by a result of Bessaga and Pełczyński [1]. Hence, (a) of Theorem 2 holds.

The proof of Theorem 2 uses further lattice analogues of the results of [10] (in particular Lemma 6 which can be deduced from the crucial lemma 1 of [10]) as well as the machinery in Banach lattice theory developed in [5], [6], and [7].

In the next section we review some notations and definitions in the theory of Banach lattices; preliminary lemmas are given in §3 and the proofs of the theorems are given in §4.

§2. In this paper we consider Banach spaces over the real field. The dual of a Banach space is denoted by E'. By an operator $T: E \to F$ between Banach spaces we mean a continuous linear map. The adjoint of an operator T is denoted by T'. An operator T is called an isomorphism or an embedding if T is injective and has closed range.

The terminology and notations for Banach lattices follow [12]. For the sake of convenience we explain frequently used terms.

Let E be a Banach lattice and $x, y \in E$ with $x \leqq y$, then the order interval $[x, y]$ is the set $\{z \in E: x \leqq z \leqq y\}$; x and y are disjoint if $|x| \wedge |y| = 0$. A linear subspace I of a Banach lattice E is called an ideal if $x \in I$ and $y \in E$ with $|y| \leqq |x|$ implies $y \in I$. An ideal I of E is called a band if $A \subset I$ and $\sup A = x \in E$ implies $x \in I$. If $0 \leqq x \in E$ then the ideal generated by x is called a principal ideal and denoted by E_x. An element $0 \leqq u \in E$ is called a quasi-interior point if the principal ideal E_u is dense in E and is called a weak order unit if E is the band generated by u.

An operator $T: E \to F$ between Banach lattices is called positive if $0 \leqq x \in E$ implies $Tx \geqq 0$, and is called an order isomorphism if $0 \leqq x$ is equivalent to $0 \leqq Tx$; T is called a lattice homomorphism if $x \wedge y = 0$ implies $Tx \wedge Ty = 0$; if, in addition, $Tx \wedge Ty = 0$ implies $x \wedge y = 0$, T is called a lattice isomorphism (a lattice isomorphism is an order isomorphism). The reader should note that lattice isomorphisms and order isomorphisms are not necessarily operator isomorphisms since their ranges are not required to be closed. A positive operator $T: E \to F$ is interval preserving (resp. almost interval preserving) if $T[0, x] = [0, Tx]$ (resp. if $T[0, x]$ is dense in $[0, Tx]$) for all $0 \leqq x \in E$. We shall make frequent use of the following fundamental fact ([6]): Let $T: E \to F$ be a positive operator; then T is almost interval preserving if and only if T' is a lattice homomorphism while T is a lattice homomorphism if and only if T' is (almost) interval preserving.

An operator T from a Banach space E into a Banach lattice F is called majorizing if T maps every (norm-) null sequence into an order interval.

If B is a band in the dual E' of a Banach lattice E then there is a unique positive contractive projection P from E' onto B with $Px' = 0$ for all $x' \in E'$ satisfying $x' \wedge y' = 0$ for all $y' \in B$; P is called the band projection onto B.

§3. In this section we collect several lemmas which will be used in §4 below for the proofs of Theorems 1 and 2. The first two lemmas are minor variations of results in [10].

LEMMA 1. *Let X be a compact Hausdorff space and let $\{\lambda_\gamma\}_{\gamma\in\Gamma}\subset C(X)'$ be a family isometrically equivalent to the usual basis of $l^1(\Gamma)$ for some infinite set Γ such that for the weak* topology the closure K of $\{\lambda_\gamma\}_{\gamma\in\Gamma}$ is dense-in-itself and metrizable. Then for all $0<\varepsilon<1$ there is a subspace F of $C(X)$ isometric to $C(\Delta)$ such that*

$$\|f\|\leqq(1-\varepsilon)^{-1}\sup_{\gamma\in\Gamma}|\langle f,\lambda_\gamma\rangle|$$

for all $f\in F$.

PROOF. At first we remark that in proposition 3 of [10] the separability of the Banach space is only needed to ensure that K is metrizable. Moreover, the index set $\mathbf{N}$ can be replaced by any infinite set Γ. The lemma now follows immediately from proposition 3 and theorem 2(b) (i) of [10].

LEMMA 2. *Let E be a separable Banach lattice with E' weakly sequentially complete and let W be a convex bounded symmetric non-norm-separable subset of E'. Then there is a $\delta>0$ such that for all $\varepsilon>0$ there exist an uncountable family $\{w'_\gamma\}_{\gamma\in\Gamma}\subset W$ and a family $\{x'_\gamma\}_{\gamma\in\Gamma}\subset E'$ of pairwise disjoint elements such that*

$$\|w'_\gamma-x'_\gamma\|\leqq\varepsilon\qquad\text{and}\qquad\|x'_\gamma\|\geqq\delta$$

for all $\gamma\in\Gamma$.

PROOF. It follows from (2.4) (c) of [5] that E' satisfies the equivalent conditions of (2.1) of [5]. In particular, each closed ideal of E' is a band and each order interval $[0,x']$ in E' is weakly compact. Since on $[0,x']$ the weak and the weak* topologies coincide the interval $[0,x']$ is weakly separable and, because of convexity, norm separable. Hence the norm closure of every principal ideal in E' is a separable band in E'. If $E=C(X)$ and $x',y'\in C(X)'$ then $L^1(|x'|)$ is the band in $C(X)'$ generated by $|x'|$ and $(dy'/dx')x'$ is the band projection of y' onto the band $L^1(|x'|)$. Now the proof is almost verbatium the same as the proof of lemma 4 in [10]; one has only to replace $L^1(|x'|)$ by the band generated by $|x'|$ and $(dy'/dx')x'$ by $P_{x'}y'$ where $P_{x'}$ is the band projection onto the closed ideal generated by x'.

The next two lemmas are well known.

LEMMA 3. *Let E be a Banach lattice and let $0<u\in E$. Then there is a compact Hausdorff space X and a lattice isomorphism $T\colon C(X)\to E$ onto the*

principal ideal E_u with $T1 = u$ and $\|T\| = \|u\|$. The adjoint T' is necessarily a lattice homomorphism and interval preserving.

PROOF. The existence of X and T follows immediately from Kakutani's Theorem on AM-spaces since the principal ideal E_u with $[-u, u]$ as unit ball is an AM-space with unit [12, (II.7.2) cor.]. Since T is a lattice homomorphism and interval preserving, T' is a lattice homomorphism and interval preserving (see §2).

LEMMA 4. *Let X be a compact Hausdorff space, let $0 < f \in C(X)$, and let $\mu, \nu \in C(X)'$ with $\mu \wedge \nu = 0$. Then there exist two sequences $(g_n), (h_n) \subset [0, f]$ with*

$$\lim_n \langle g_n, \mu \rangle = \langle f, \mu \rangle,$$

$$\lim_n \langle h_n, \nu \rangle = \langle f, \nu \rangle$$

and, for all $n \in \mathbf{N}$,

$$\operatorname{supp} g_n \cap \operatorname{supp} h_n = \emptyset.$$

PROOF. Since μ and ν are positive and disjoint $\mu + \nu = |\mu - \nu|$. It follows now from (II.4.2) cor. 1 of [12] that

$$\begin{aligned}\langle f, \mu + \nu \rangle &= \sup_{|\tilde{f}| \leqq f} \langle \tilde{f}, \mu - \nu \rangle \\ &\leqq \sup_{|\tilde{f}| \leqq f} (\langle \tilde{f}^+, \mu \rangle + \langle \tilde{f}^-, \nu \rangle) \\ &\leqq \langle f, \mu + \nu \rangle.\end{aligned}$$

Hence there is a sequence $(f_n) \subset [-f, f]$ with $\lim_n \langle f_n^+, \mu \rangle = \langle f, \mu \rangle$ and $\lim_n \langle f_n^-, \nu \rangle = \langle f, \nu \rangle$. By putting $g_n = (f_n^+ - n^{-1}1)^+$ and $h_n = (f_n^- - n^{-1}1)^+$ the sequences (g_n) and (h_n) have the desired properties.

Our next lemma shows that (a) of Theorem 2 can be replaced by the condition that there exists a non-zero lattice homomorphism from L^1 into E'.

LEMMA 5. *Let E be an AL-space and let T be a lattice homomorphism from E into a Banach lattice F with $\|T\| = 1$. Then for every $\varepsilon > 0$ there exists a band $B \subset E$, $B \neq \{0\}$, such that $\|Tx\| \geqq (1 - \varepsilon)\|x\|$ for all $x \in B$.*

PROOF. Choose $0 < u \in E$ with $\|u\| = 1$ and $\|Tu\| > 1 - \varepsilon$. Let $B_1 = \overline{E_u}$ and $T_1 = T_{|B_1}$. Since u is a weak order unit in B_1, by Kakutani's Theorem [12, II. 8.5] B_1 is norm and lattice isomorphic to a space $L^1(X, \Sigma, \mu)$, μ finite. Since

$\|T_1\| > 1 - \varepsilon$ there exists $0 < y' \in F'$ with $\|y'\| = 1$ and $\|T_1'y'\| > 1 - \varepsilon$. Hence there exists a set $A \in \Sigma$, $\mu(A) > 0$ such that $(T_1'y')(t) \geqq 1 - \varepsilon$ for all $t \in A$. Now let B be the band of all equivalence classes of functions vanishing outside A μ-a.e. Then for all $f \in B$,

$$\|T_1 f\| = \|T_1|f|\| \geqq \langle |f|, T_1'y' \rangle \geqq (1 - \varepsilon)\|f\|.$$

Since B_1 is a band in E, B is also a band in E with $\|Tf\| \geqq (1 - \varepsilon)\|f\|$. Since $\mu(A) > 0$, it follows that $B \neq \{0\}$.

Following Pełczyński [8], a sequence (A_i) of subsets of a set A is called a tree if $A_{2i} \cap A_{2i+1} = \varnothing$ and $A_{2i} \cup A_{2i+1} \subset A_i$ for all $i \in \mathbf{N}$. In the sequel we denote the set of all Lebesque measurable subsets of $[0,1]$ with non-vanishing measure by Σ_+.

Lemma 6. *Let E be a Banach lattice such that there exists a non-zero lattice homomorphism from L^1 into E'. Then given $\varepsilon > 0$, there exists a compact Hausdorff space X, a tree (K_i) of compact sets in X, a lattice isomorphism $U: C(X) \to E$ onto an ideal of E with $\|U\| = 1$, and a sequence (x_i') of positive elements in E' with $\|x_i'\| \leqq 1 + \varepsilon$ such that for all i, $U'x_i'$ is a probability measure on X with* $\operatorname{supp} U'x_i' \subset K_i$.

Proof. We give two arguments. The first is a fairly quick deduction from the results of [10].

Assume $\varepsilon < 1$. Let $R: L^1 \to E'$ be a lattice homomorphism with $\|R\| = 1$. Choose $0 < u \in E$ with $\|u\| = 1$ such that $\|R'u\| > (1 + \frac{1}{3}\varepsilon)^{-1}$ when u is considered as an element of E''. By Lemma 3 there exists a compact Hausdorff space X and a lattice isomorphism $U: C(X) \to E$ with $U1 = u$, $\|U\| = 1$, and such that U' is a lattice homomorphism and interval preserving. Then $U'R$ is a lattice homomorphism with $\|U'R\| > (1 + \frac{1}{3}\varepsilon)^{-1}$. By Lemma 5 there exists a band $B \subset L^1$, $B \neq \{0\}$ such that $G = U'RB$ is a closed sublattice of $C(X)'$ lattice isomorphic to L^1 and thus isometric to L^1 with $\|(U'R_{|B})^{-1}\| \leqq 1 + \varepsilon/3$. By the known structure of such sublattices (see proposition 2 of [10]), there exists a regular Borel probability measure μ on X and a σ-algebra $\mathfrak{F}$ of Borel subsets of X such that $(X, \mathfrak{F}, \mu \mid \mathfrak{F})$ is a purely non-atomic measure space and $G = L^1(\mu \mid \mathfrak{F})$. (The point of the correction to [10] is that a general subspace of $C(X)'$ isometric to L^1 has a slightly more complicated structure; however the argument for proposition 2 of [10] is indeed valid for sublattices of $C(X)'$.) Then by lemma 1 of [10] there exists a tree (K_i) of compact sets in X and a tree (F_i) of elements of $\mathfrak{F}$ so that $\mu(K_i) > 0$, $\mu(F_i) \leqq (1 + \frac{1}{3}\varepsilon)\, \mu(K_i)$ and $K_i \subset F_i$ for all $i \in \mathbf{N}$. Thus fixing i, $[\mu(K_i)]^{-1}\chi_{F_i}$ is a positive element of G of norm at most $(1 + \frac{1}{3}\varepsilon)$. Since

$\|(U'R_{|B})^{-1}\| \leqq 1+\frac{1}{3}\varepsilon$, there exists a positive element $y'_i \in E'$ with $\|y'_i\| \leqq (1+\frac{1}{3}\varepsilon)^2 \leqq 1+\varepsilon$ and $U'y'_i = [\mu(K_i)]^{-1}\chi_{F_i}$. Since U' is interval preserving, there exists an element $x'_i \in E$ with $0 \leqq x'_i \leqq y'_i$ and $U'x'_i = [\mu(K_i)^{-1}]\chi_{K_i}$. Since $\|x'_i\| \leqq \|y'_i\| \leqq 1+\varepsilon$, this completes the first argument.

An alternate and self contained argument, using purely lattice theoretic methods, is as follows:

Let $R: L^1 \to E'$ be a lattice homomorphism with $\|R\| = 1+\varepsilon$ and let $'R = R'_{|E}$. Then there exists $0 < u \in E$ with $\|u\| = 1$ and $\|'R(u)\| > 1+\frac{1}{2}\varepsilon$. By Lemma 3 there exist a compact space X and a lattice isomorphism $U: C(X) \to E$ with $U1 = u$ and $\|U\| = 1$ such that U' is a lattice homomorphism and is interval preserving. Thus $U'R$ is also a lattice homomorphism. If we denote $'R \circ U$ by T then $\int_A Tf\,dt = \langle f, U'R\chi_A \rangle$ for all $f \in C(X)$ and all $A \in \Sigma_+$.

Now choose a sequence (ε_i) in $\mathbf{R}$ with $0 < \varepsilon_i$, and $\varepsilon_{2i}, \varepsilon_{2i+1} < \varepsilon_i$ for all $i \in \mathbf{N}$. We shall construct a sequence (f_i) in $C(X)$, $0 \leqq f_i \leqq 1$, and a tree (A_i) of sets in $[0,1]$, $A_i \in \Sigma_+$, such that the sequence (K_i) with $K_i = \operatorname{supp} f_i$ forms a tree of sets in X and such that $(Tf_i)(t) > 1+\varepsilon_i$ for all $t \in A_i$. Let $f_1 = 1$. Since $\|T1\| = \|'Ru\| > 1+\frac{1}{2}\varepsilon$ there is a set $A_1 \in \Sigma_+$ such that $(Tf_1)(t) = (T1)(t) > 1+\frac{1}{2}\varepsilon$ for all $t \in A_1$. Suppose f_i and A_i have been constructed with $(Tf_i)(t) > 1+\varepsilon_i$ for all $t \in A_i$. Choose $B, C \in \Sigma_+$ with $B \cap C = \emptyset$ and $B \cup C \subset A_i$. Then $U'R\chi_B \wedge U'R\chi_C = 0$. Hence by Lemma 4 there exist two sequences (g_n) and (h_n) of positive elements with $g_n + h_n \leqq f_i$ and $\operatorname{supp} g_n \cap \operatorname{supp} h_n = \emptyset$ Then *all* $n \in \mathbf{N}$, such that $\lim_n \langle g_n, U'R\chi_B \rangle = \langle f_i, U'R\chi_B \rangle$ and $\lim_n \langle h_n, U'R\chi_C \rangle = \langle f_i, U'R\chi_C \rangle$, or equivalently, such that $\lim_n \int_B Tg_n\,dt = \int_B Tf_i\,dt$ and $\lim_n \int_C Th_n\,dt = \int_C Tf_i\,dt$. Since $0 \leqq Tg_n \leqq Tf_i$, and $0 \leqq Th_n \leqq Tf_i$ there exist subsequences (g_{n_k}) and (h_{n_k}) such that the sequence (Tg_{n_k}) (resp. (Th_{n_k})) converges almost uniformly on B (resp. C) to Tf_i (Egoroff's Theorem). Hence there exist two positive functions f_{2i} and f_{2i+1} in $C(X)$ and sets A_{2i} and A_{2i+1} in Σ_+, $A_{2i} \subset B$, $A_{2i+1} \subset C$, such that $f_{2i+1} \leqq f_i$, $\operatorname{supp} f_{2i} \cap \operatorname{supp} f_{2i+1} = \emptyset$ and $(Tf_{2i})(t) > 1+\varepsilon_{2i}$ for all $t \in A_{2i}$ and $(Tf_{2i+1})(t) > 1+\varepsilon_{2i+1}$ for all $t \in A_{2i+1}$. This completes the construction of the f_i's and the A_i's.

Now let $y'_i = R(\|\chi_{A_i}\|^{-1}\chi_{A_i})$. Then $\|y'_i\| \leqq \|R\| = 1+\varepsilon$, and $\langle f_i, U'y_i \rangle > 1$. Let $K_i = \operatorname{supp} f_i$. Since U' is interval preserving there exist elements $z'_i \in [0, y'_i]$ such that $U'z'_i = \chi_{K_i}U'y'_i$. Clearly $\langle f_i, U'z'_i \rangle \geqq 1$ and thus $\|U'z'_i\| \geqq 1$. Let $x_i = \|U'z'_i\|^{-1}z'_i$. Then X, U, (x'_i) and (K_i) have the desired properties which concludes the proof.

§4. In this section we present the proofs of our main results.

Proof of Theorem 1. It suffices to show that E contains a subspace G

isomorphic to $C(\Delta)$ such that $T_{|G}$ is an isomorphism since by theorem 1 of [9] G contains a subspace G_0 isomorphic to $C(\Delta)$ and complemented in E. Let V^0 be the unit ball of F' and let $W = T'V^0$. Then W satisfies the hypotheses of Lemma 2. Now choose $\delta > 0$ as in Lemma 2 and for $\varepsilon > 0$ with $\delta(1-\varepsilon)^2 - \varepsilon > 0$ choose $\{w'_\gamma\}_{\gamma\in\Gamma}$ and $\{x'_\gamma\}_{\gamma\in\Gamma}$ as in Lemma 2. Since E is separable there exist $u \in E$, $\|u\| = 1$, and an uncountable subset $\Gamma_1 \subset \Gamma$ such that $1-\varepsilon < \langle u, x'_\gamma/\|x'_\gamma\|\rangle$ for all $\gamma \in \Gamma_1$. It follows from Lemma 3 that there exist a compact space X and an operator $S: C(X) \to E$ with S' a lattice homomorphism and $S[-1,1] = [-|u|,|u|]$. Then putting $\mu_\gamma = S'x'_\gamma/\|x'_\gamma\|$, $\|\mu_\gamma\| > 1-\varepsilon$ for all $\gamma \in \Gamma_1$, since there is an $f \in C(X)$ with $\|f\| \leqq 1$ and $Sf = u$. By passing to an uncountable subset $\Gamma_2 \subset \Gamma_1$ we may assume that the family $\{x'_\gamma/\|\mu_\gamma\|\|x'_\gamma\|\}_{\gamma\in\Gamma_2}$ is dense-in-itself for the weak* topology. Since S' is a lattice homomorphism the elements of $\{\mu_\gamma/\|\mu_\gamma\|\}_{\gamma\in\Gamma_2}$ are pairwise disjoint and thus isometrically equivalent to the basis of $l^1(\Gamma_2)$. Hence by Lemma 1 there is a subspace H of $C(X)$ isomorphic to $C(\Delta)$ such that for all $h \in H$,

$$\sup_{\gamma\in\Gamma_2}\left|\left\langle h, \frac{\mu_\gamma}{\|\mu_\gamma\|}\right\rangle\right| \geqq (1-\varepsilon)\|h\|.$$

This implies that for all $h \in H$

$$\sup_{\gamma\in\Gamma_2}\left|\left\langle h, \frac{S'x'_\gamma}{\|x'_\gamma\|}\right\rangle\right| \geqq (1-\varepsilon)^2\|h\|.$$

$$\sup_{\gamma\in\Gamma_2}|\langle Sh, x'_\gamma\rangle| \geqq \delta(1-\varepsilon)^2\|h\|,$$

$$\sup_{\gamma\in\Gamma_2}|\langle Sh, w'_\gamma\rangle| \geqq (\delta(1-\varepsilon)^2 - \varepsilon)\|h\|,$$

and finally,

$$\sup_{y'\in V^0}|\langle TSh, y'\rangle| \geqq (\delta(1-\varepsilon)^2 - \varepsilon)\|h\|.$$

Hence the restriction of TS to H is an isomorphism, and thus $G = SH$ is isomorphic to $C(\Delta)$ and $T_{|G}$ is an isomorphism.

Proof of Theorem 2. (c) $\Rightarrow$ (d): If (c) holds then $S \circ T$ is an isometry since φ is onto, and thus T is an embedding and clearly positive.

(d) $\Rightarrow$ (e): If T is as in (d) then $[0, T1]$ is not weakly sequentially precompact.

(e) $\Rightarrow$ (f): If $[0,x]$ is not weakly sequentially precompact, it follows from the main result of [11] that there exists a sequence $(x_i) \subset [0,x]$ equivalent to the usual l^1-basis. The corresponding embedding T of l^1 in E is clearly positive and is majorizing since T maps the unit ball of l^1 into $[-x,x]$.

(f) $\Rightarrow$ (g): Let T: $l^1 \to E$ be a majorizing embedding. Let F be any closed sublattice of E containing the range of T. It follows from (IV. 3.4.d) of [12] that T considered as an operator from l^1 into F is majorizing and that T factors through an AM-space M, $l^1 \xrightarrow{R} M \xrightarrow{S} F$ with S positive. Now assume that F' has a weak order unit, say w'. Since F' is the band generated by w' and since S', as the adjoint of a positive operator, is order continuous, $S'F'$ is contained in the band B of M' generated by $S'w'$. Since M' is an AL-space, B is the closed linear hull of the weakly compact order interval $[0, S'w']$, i.e. B is weakly compactly generated. Now T' is onto and thus $R'B = l^\infty$. Hence l^∞ is also weakly compactly generated which is absurd. Therefore, F' cannot have a weak order unit. By taking for F the closed sublattice generated by the range of T one gets a separable sublattice such that F' has no weak order unit.

(g) $\Rightarrow$ (h): Let F be a separable closed sublattice of E and let u be a quasi-interior point of F [12, II. 6.2]. If F' does not have a weak order unit then there exists in F' a normalized family $\{y'_\gamma\}_{\gamma \in \Gamma_1}$, Γ_1 uncountable, of positive pairwise disjoint elements. Since $\langle u, y'_\gamma \rangle > 0$ for all $\gamma \in \Gamma_1$ there exist a $\delta > 0$ and an uncountable subset $\Gamma \subset \Gamma_1$ such that $\langle u, y'_\gamma \rangle > \delta$ for all $\gamma \in \Gamma$. It is easily seen that the closed linear hull of $\{y'_\gamma\}_{\gamma \in \Gamma}$ is a sublattice of F' lattice isomorphic to $l^1(\Gamma)$.

(h) $\Rightarrow$ (i): Let F be as in (h) and choose a quasi-interior point $u \in F$. A standard argument shows the existence of an infinite bounded set $A \subset F'$ of positive pairwise disjoint elements, dense-in-itself for the weak* topology, with $\langle u, y' \rangle = 1$ for all $y' \in A$. Let B be the weak* closure of A. Since F is separable, B is weak* metrizable. Now choose by Lemma 3 a compact space X and a lattice isomorphism R from $C(X)$ onto the ideal F_u with $R1 = u$. Since F_u is dense in F, R' is a lattice isomorphism and its restriction is a homeomorphism of the weak* compact set onto the weak* compact set $K = R'B$; in particular, K is also weak* metrizable. The family $\{R'y'\colon y' \in A\} \subset C(X)'$ is dense-in-itself for the weak* topology and isometrically equivalent to the usual basis of $l^1(A)$. Now, as in the proof of proposition 3 of [10] it follows that there is a weak* compact set $Z \subset K$ homeomorphic to the Cantor set Δ such that the canonical map S: $C(X) \to C(Z)$ defined by $(Sf)(\mu) = \langle f, \mu \rangle$ for all $\mu \in Z$, $f \in C(X)$ has an adjoint S' which is an isometry. (Observe that the separability assumption in proposition 3 of [10] is only required to ensure the metrizability of K for the weak* topology.) Since S is a positive operator, S' is a positive isometry from an AL-space into another AL-space and hence a lattice isomorphism. Let Y be the inverse image $S'^{-1}Z \subset B$. Then Y is also homeomorphic to Δ. Let T: $F \to C(Y)$

be the operator defined by $(Tx)(y') = \langle x, y' \rangle$ for all $x \in F$, $y' \in Y$. By identifying $C(Z)$ and $C(Y)$ canonically with $C(\Delta)$ it follows that $S = T \circ R$. Since S is onto, T is onto and clearly positive. Since R' and S' are lattice isomorphisms T' is a lattice isomorphism and hence T is almost interval preserving (see §2). q.e.d.

(i) $\Rightarrow$ (b): Let F be a closed sublattice of E and let T be an almost interval preserving operator from F onto $C(\Delta)$. Then T' is a lattice isomorphism and an embedding of $C(\Delta)'$ in F'. It follows that $C(\Delta)'$ is also lattice isomorphic to a closed sublattice of E' (see the proof of Theorem 2, (e) $\Rightarrow$ (a) in [7]).

(b) $\Rightarrow$ (a): This is trivial.

(a) $\Rightarrow$ (c): If $\varepsilon > 0$ is given choose X, U: $C(X) \to E$, $(x_i') \subset E'$, and (K_i) as in Lemma 6. Using Urysohn's Lemma one can choose a sequence $(f_i) \subset C(X)$ of positive functions with $f_1 = 1$, $f_{2i} + f_{2i+1} = f_i$ and $f_i(K_i) = \{1\}$ for all $i \in \mathbf{N}$. Now let (g_i) be a total sequence of positive characteristic functions in $C(\Delta)$ with $g_1 = 1$, $g_{2i} + g_{2i+1} = g_i$, and $g_{2i} \wedge g_{2i+1} = 0$ for all $i \in \mathbf{N}$. Then there exists an operator V: $C(\Delta) \to C(X)$ with $Vg_i = f_i$ for all $i \in \mathbf{N}$. Clearly, V is an isometry and an order isomorphism. Hence, $T = U \circ V$ is an order isomorphism with $\|T\| = 1$. Now let (i_n) be a sequence in $\mathbf{N}$ with $i_{n+1} \in \{2i_n, 2i_n + 1\}$ for all $n \in \mathbf{N}$ and let x' be a cluster point of the sequence (x_{i_n}'). Then $0 \leqq x'$ and $\|x'\| \leqq 1 + \varepsilon$. It is readily verified that $T'x'$ is a Dirac measure on Δ and that every Dirac measure on Δ can be obtained in this way. Now let $K = \{x' \in E' \colon 0 \leqq x',\ \|x'\| \leqq 1 + \varepsilon,\ T'x' \text{ is a Dirac measure on } \Delta\}$. Clearly, K is weak* compact. Denote the restriction of T' on K by φ. Then preceding considerations show that φ is a continuous map from K onto Δ (identify Δ with its canonical image in $C(\Delta)'$). Finally, define S: $E \to C(K)$ by $(Sx)(x') = \langle x, x' \rangle$ for all $x \in E$, $x' \in K$. Then K, φ, T, and S have the desired properties.

This concludes the proof that (a)–(i) are equivalent.

Now assume that E is a separable Banach lattice. It follows from the proof of (f) $\Rightarrow$ (g) $\Rightarrow$ (h) $\Rightarrow$ (i) that one can now choose $F = E$. The implication (j) $\Rightarrow$ (c) is trivial. To conclude the proof of Theorem 2 we show (a) $\Rightarrow$ (j): Choose K, φ, T and S as in the proof of (a) $\Rightarrow$ (c). Since E is separable, K is metrizable. Since Δ is uncountable it follows from a result of Kuratowski ([4], p. 351) that there is a compact set $K_1 \subset K$ homeomorphic to Δ such that the restriction $\varphi_{|K_1}$ is a homeomorphism. It follows from a result of Sierpinski ([13], p. 118) that there is a retract ψ from Δ onto $\varphi(K_1)$. Denote the canonical injection of K_1 onto K by χ. Then $\psi \circ \varphi \circ \chi$ is a homeomorphism from K_1 onto $\varphi(K_1)$. If one replaces in (c) K by K_1, φ by $\psi \circ \varphi \circ \chi$, T by $\bar{T}$ with $\bar{T}f = T(f \circ \psi)$ for all $f \in C(\varphi(K_1))$, and S by $\bar{S}$ with $\bar{S}x = (Sx)_{|K_1}$ for all $x \in E$, one gets (j). This completes the proof of Theorem 2.

REMARK. The application of Kuratowski's Theorem in the proof of (a) $\Rightarrow$ (j) is similar to the one made by Pełczyński in [9], where it is shown that if E is a separable Banach space and F is a closed subspace of E isomorphic to $C(\Delta)$ then there is a closed subspace $G \subset F$ isomorphic to $C(\Delta)$ and complemented in E. A particularly elegant proof of Pełczyński's Theorem may be found in [2].

We raise the following question: Suppose $C(\Delta)$ is isomorphic to a closed subspace of a Banach lattice E. Do the equivalent conditions of Theorem 2 hold?

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